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The Green function and correlation exponents for interacting particles in one dimension

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Abstract. We give a heuristic argument that shows how all the correlation functions at large distances and times of a translationally invariant one-dimensional quantum liquid are determined by one (bosons or spinless fermions) or two (spin-1/2 fermions) correlation exponents. The argument is based on the observation that in the path-integral formulation of the problem the effect of creation and destruction operators can be seen on inserting dislocations in a two-dimensional crystal of particle world lines.

We consider a translationally invariant one-dimensional interacting quantum systems of particles. According to the Feynman path-integral formalism [1], the statistical properties of the wavefunction as functions of position x and imaginary time t will be the same as those of an ensemble of world lines at temperature $\hbar$.

For *bosons* or *spinless fermions* the macroscopic (long-wavelength) behaviour is described by an action of the form

$$A_{\text{spinless}} = \tfrac{1}{2}\int \mathrm{d}x\,\mathrm{d}t\left(\mu\dot{u}^2 + Ku'^2\right) \tag{1}$$

where $u(x, t)$ is a field describing the displacement of the particles away from a uniform distribution. The two terms represent the kinetic energy (where μ is the effective mass density) and an elastic response due to the interparticle interactions (where K is the bulk compressibility). We note that in one dimension the distinction between bosons and spinless fermions is not as significant as it is in higher dimensionality, because we cannot go from a configuration to one with exchanged particles without having two particles at the same point in space at some intermediate stage. If we restrict ourselves to the sector in which the particles are in order along the line ($x_1 \leqslant x_2 \leqslant \ldots \leqslant x_M$), the Pauli exclusion principle reduces to a boundary condition specifying the vanishing of the wavefunction whenever two particles are at the same coordinate. It follows that for any interacting fermion problem there is a corresponding interacting boson problem which has the same action in the long-wavelength limit, and thus many of the same physical properties (e.g. free fermions correspond to hard-core point bosons). The action (1) with *renormalized* μ and K also describes a quantum many-body system placed in an external potential if the latter is irrelevant in the renormalization group sense.

For *fermions with spin* we could attempt a similar representation in terms of two fields $u_\uparrow$ and $u_\downarrow$ for the two spin populations. However, the spin-up and spin-down electrons are

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physically identical and have the same interactions with particles of the same spin; then the interparticle interaction between the two degenerate systems is never a small perturbation, and the proper eigenstates are the sum and difference of the two fields [2, 3]. Thus the natural variables for the problem are the mass (or charge) density $C = (u_\uparrow + u_\downarrow)/2$ and spin density $S = (u_\uparrow - u_\downarrow)/2$. For a translationally invariant system this separation of spin and charge is a rigorous property. An external periodic potential can destroy this separation, notably in the cases when the spin ordering is incommensurate with the periodicity of the potential [4, 5], or when the interaction between particles of the same spin is very strong [3]; however, in many cases of interest the spin and charge fields are uncoupled, and the original problem is split into two simpler problems:

$$A_{\text{spin-1/2 Fermi}} = \tfrac{1}{2}\int \mathrm{d}x\,\mathrm{d}t\Big(\mu_C\dot{C}^2 + K_C C'^2 + \mu_S\dot{S}^2 + K_S S'^2\Big). \tag{2}$$

The interpretation of the coefficients parallels those of the spinless case: K_C and K_S describe the response to changes in density and total spin (and thus can be interpreted as the compressibility and magnetic susceptibility); μ_C and μ_S are the dynamic coefficients for the two fields and can be thought of as mass densities. In writing equation (2) we assumed that neither the external potential nor the interaction between electrons of opposite spin produce energy gaps in charge or spin excitation spectra.

The field theory expressed in (1) and (2) starts from elasticity theory; the description is in terms of the sound and spin waves of the system. This might be regarded as an unusual place to start; certainly in three dimensions we would expect that single-particle degrees of freedom were more important, and indeed the theory for that case is written that way. However, in the present case we can give two arguments why these are adequate models and appropriate long-wavelength descriptions.

(1) A single-particle excitation is (to a first approximation) a motion of a single particle, ignoring the presence of the others. In higher dimensions the particles may or may not be able to move around each other, and then we can have liquids as well as solids. However, in one dimension the particles cannot freely move past each other, and as a result the possible condensed phases are closer to being non-crystalline solids.

(2) All systems have longitudinal sound waves. In higher-dimensional systems this description is incomplete whenever transverse phonon modes are absent, as they are in fluids and gases. In one dimension, however, the compressional modes necessarily exist, and these exhaust the one degree of freedom that each particle has; the single-particle motions are already implicitly included in our description.

A great deal of the description of the interacting quantum fluid can be given in terms of the parameter g:

$$g = \frac{\pi\hbar n^2}{\sqrt{\mu K}} \tag{3}$$

for the action (1) (with one internal degree of freedom), with n being the particle density, and the parameters g_c and g_s for the system with spin

$$g_C = \frac{\pi\hbar(2n)^2}{\sqrt{\mu_C K_C}} \qquad g_S = \frac{\pi\hbar(2n)^2}{\sqrt{\mu_S K_S}}. \tag{4}$$

The parameter $2n$ that appears in (4) is the total particle density. With the bare μ and K these characterize the degree of quantum fluctuations in the spin and charge subsystems. For sufficiently large g_ν, the external potential and the interaction between particles of the opposite spin are irrelevant in the renormalization sense, which means that the spin and

charge degrees of freedom are not localized. The other relevant combinations of the μ and K for a gapless system are the sound velocities

$$c_\nu^2 = K_\nu/\mu_\nu \qquad \nu = c, s. \tag{5}$$

The parameters g_ν computed with the *renormalized* μ_ν and K_ν (i.e. macroscopic and measurable, as opposed to the microscopic values) determine the long-distance behaviour of all the correlation functions [6–11]. The purpose of the present paper is to give a simple construction for determining which combination of the g-values appears.

The simplest case is the Green function for bosons. The quantity that we wish to calculate is

$$G(x, t) = \left\langle \psi^\dagger(x, t)\psi(0, 0)\right\rangle \tag{6}$$

where $\psi^\dagger(x, t)$ is the operator that creates a particle. From the viewpoint of the path-integral formalism that we are using, we must imagine a field of world lines running in the t direction, and the expectation value in equation (6) measures the average amplitude for having a world line that ends at (0, 0), and another that begins at (x, t), within an ensemble with weights determined by (1).

Of course, the very first problem that we must address is that, in the original ensemble, lines cannot end at all. This is simply solved by adding a term to the action that permits this possibility with a very large energy E_{core}, so that the density of line breaks will be extremely small. Then the Green function (6) can be interpreted as the probability that having found a line end at the origin there is a simultaneous line beginning at distance x from it, as shown in figure 1.

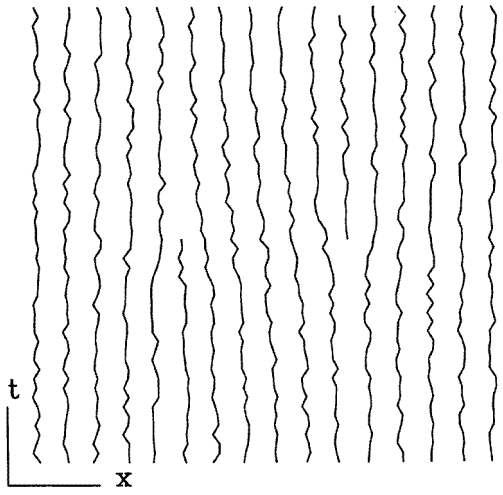

Figure 1. Dislocation pair in a two-dimensional crystal of world lines. This shows the configuration of the particle world lines in one member of the ensemble. A particle is removed at $(x = 0, t = 0)$, terminating a line, and another particle is added elsewhere, giving rise to another half-line. The wandering of the lines represents the quantum fluctuations.

In the two-dimensional crystal of world lines the line ends are oppositely directed dislocations, which give rise to a distortion in the field u which has a Burger vector n^{-1} (because one world line has disappeared); the distortion falls off as r^{-1} and thus gives a

logarithmic energy (logarithmic action, in the quantum problem) [12]

$$E_{\text{dislocation pair}} = \frac{\sqrt{\mu K}}{2\pi n^2}\ \ln(nR) + E_{\text{core}} \equiv \frac{\overline{h}}{2g}\ \ln(nR) + E_{\text{core}} \tag{7}$$

where $R = \sqrt{x^2 + c^2t^2}$ is the length coordinate in isotropic units $(x,\ ct)$. The core energy E_{core} is large, so that the density of dislocations is small; in fact we shall assume that there is just one pair present. The wavefunction for the quantum problem is the partition function at 'temperature' $\overline{h}$ for the world-line crystal, and so the probability that they are found at distance R from each other is then proportional to

$$\text{Prob}(R) \propto \exp\bigl(-E_{\text{dislocation pair}}/\overline{h}\bigr) \propto (nR)^{-1/2g}. \tag{8}$$

This gives the leading term in the scaling behaviour of the Green function for a bosonic system.

The Green function for a spinless fermionic system can be calculated in a similar way. The argument leading to (8) continues to apply; however, the new feature here is that the fermionic wavefunction changes sign as particles are moved past each other, so that we must append to (8) a sign factor

$$\text{Sign}(R) \approx \langle\cos\{\pi n[x + u(x,\ t) - u(0,0)]\}\rangle. \tag{9}$$

Here nx is the number of world lines in the interval x in a perfect crystal, and $u(x,\ t) - u(0,0)$ is a correction for the quantum fluctuations in the locations of the lines. The average is over the 'thermal' ensemble with Boltzmann weight $A_{\text{spinless}}/\overline{h}$ (1); in writing this expression it is assumed that there is no long-range order in the line positions, so that the correlation function is translationally invariant. The effect of the fluctuations is to decrease the expected phase factor $\cos(\pi nx)$ by a Debye–Waller factor $\exp\{-(\pi n)^2\langle[u(x,\ t) - u(0,0)]^2\rangle/2\}$; in two dimensions, $(\pi n)^2\langle[u(x,\ t) - u(0,0)]^2\rangle/2$ is given by $(g/2)\ \ln(nR)$, with the result that

$$G(x,\ t) \propto \text{Prob(R)}\ \text{Sign}(R) \approx (nR)^{-(g/2+1/2g)}\ \cos(\pi nx). \tag{10}$$

The Fourier transform of the equal-time Green function gives the single-particle momentum distribution $\langle n_k\rangle$, and near $k_F = \pi n$ it has the form

$$\langle n_k\rangle = \text{constant} - \text{Sign}\bigl(k - k_F\bigr)\bigl|k - k_F\bigr|^{\alpha} \tag{11}$$

where $\alpha = g/2 + 1/2g - 1$. This shows that there will not be a sharp Fermi surface in one dimension, except for the special case $g = 1$ (which includes the case of non-interacting particles).

The argument can be generalized to the case of an arbitrary correlation function. Consider the four-point function

$$C\bigl(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4\bigr) = \Bigl\langle\psi^{\dagger}\bigl(\boldsymbol{x}_1\bigr)\psi^{\dagger}\bigl(\boldsymbol{x}_2\bigr)\psi\bigl(\boldsymbol{x}_3\bigr)\psi\bigl(\boldsymbol{x}_4\bigr)\Bigr\rangle \tag{12}$$

where $\boldsymbol{x}_i = (x_i,\ ct_i)$ is the representation in which the action (1) is isotropic. Each operator creates a dislocation (or a dislocation in each field, for particles with spin); the dislocations interact as two-dimensional Coulomb charges, with an energy that depends logarithmically on the distances $R_{ij} = \sqrt{(x_i - x_j)^2 + c^2(t_i - t_j)^2}$ with a pre-factor $\overline{h}/2g$ (for pairs of dislocations of the same kind, corresponding to operators $\psi\psi$ or $\psi^{\dagger}\psi^{\dagger}$) or $-\overline{h}/2g$ (for operator pairs $\psi^{\dagger}\psi$). The four-point function generalizing the probability function (8) is then given by the product of the resulting factors $R_{ij}^{\pm 1/2g}$:

$$\text{Prob}\bigl(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4\bigr) = \left[\frac{n^{-2}R_{12}R_{34}}{R_{13}R_{14}R_{23}R_{24}}\right]^{1/2g}. \tag{13}$$

The correlation function will have a phase factor, which is constructed by averaging the product of phase factors for each operator:

$$\mathrm{Sign}(x_1, x_2, x_3, x_4) = \langle \phi(x_1 + u_1)\phi(x_2 + u_2)\phi(x_3 + u_3)\phi(x_4 + u_4)\rangle \quad (14)$$

where the average is over the Gaussian variables $u_i = u(x_i, t_i)$, and

$$\phi(x) = \begin{cases} \cos(\pi n x) & \text{for fermions} \\ A + B\ \cos(2\pi n x) & \text{for bosons} \end{cases} \quad (15)$$

represents the leading Fourier components of the spatial variation in the phase of the wavefunction. Frequently it is appropriate to keep just the leading (constant) term in ϕ for the bosonic case, and then Prob() will be the leading term in the four-point correlation function for bosons. With the assumption that there is no long-range order in the line positions, only the translationally invariant combinations of the x_i survive in the averaging; for the fermionic four-point function the result is

$$\begin{aligned} \mathrm{Sign}() \approx & \left[\frac{n^{-2}R_{12}R_{34}}{R_{13}R_{14}R_{23}R_{24}}\right]^{g/2} \cos\left[\pi n(x_1 + x_2 - x_3 - x_4)\right] \\ & + \left[\frac{n^{-2}R_{14}R_{23}}{R_{12}R_{13}R_{24}R_{34}}\right]^{g/2} \cos\left[\pi n(x_1 - x_2 - x_3 + x_4)\right] \\ & + \left[\frac{n^{-2}R_{13}R_{24}}{R_{12}R_{14}R_{23}R_{34}}\right]^{g/2} \cos\left[\pi n(x_1 - x_2 + x_3 - x_4)\right] \end{aligned} \quad (16)$$

and the leading term in the fermionic four-point function is $C = \mathrm{Prob}() \times \mathrm{Sign}()$. For the special case $x_1 \approx x_3$, $x_2 \approx x_4$ the four-point function reduces to the density–density correlation function, as it should; Prob() (13) and the first two terms of (16) are constants (in carrying the large-R estimate to short distances we put $nR_{13} \approx nR_{24} \approx 1$) giving

$$\langle n(x_1)n(x_2)\rangle \propto \text{ constant} + \frac{\cos[2\pi n(x_1 - x_2)]}{[(x_1 - x_2)^2 + c^2(t_1 - t_2)^2]^g}. \quad (17)$$

This result also holds for bosons, but now the non-trivial spatial variation in the boson phase must be considered to obtain the second term of (17).

The further generalization of this argument to particles with spin is straightforward when spin and charge are uncoupled. Then the creation and destruction operators generate dislocations in both the C and the S fields, and the exponents for the decay of the Green function are the sum of the contributions from the two sectors independently; for fermions with spin the result is $\alpha = g_C/8 + g_S/8 + g_C^{-1}/2 + g_S^{-1}/2 - 1$. For the free-fermion case ($g_C = g_S = 2$) this again gives $\alpha = 0$.

This is not a rigorous derivation, since we have put in the effect of statistics by hand; however, it is possible to represent the field operators in terms of bosonic operators [13–17] and thereby to reproduce the results of this work more laboriously. Our argument reproduces the results obtained using standard methods [18].

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